The Estimation of Thermal Diffusivity through the use of Mathematical Modeling

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INTRODUCTION

The rate at which heat travels through a given substance varies according to the density of the substance, the specific heat of the substance, and the thermal conductivity of the substance. These variables can be related to each other by combining them into a ratio. This ratio is called thermal diffusivity, and can be viewed as the rate at which a thermal wave will propagate through the given substance. Soil is not a homogeneous substance as it is composed of particles with a specific or varying amount of space between the particles. This space is defined as pore space and can be measured and defined by a percentage. Therefore, a soil with 60% pore space will have a larger distance between soil particles than a soil with 40% pore space.

The pore space in a given soil can be filled with air, water, or any mixture of air and water. This fact can cause the density of the given soil to vary greatly. This fact also can cause the specific heat and the thermal conductivity to vary greatly. The overall effect of the moisture content of the soil causes the thermal diffusivity to vary accordingly. In Table I. the densities, specific heats, thermal conductivities, and thermal diffusivities of air, water, and various soil types are cited.

Table I.

Material	Condition	Mass Density (kg/m3 ?10³)	Specific Heat (J/kg·K*10³)	Heat Capacity (J/m³·K*10⁶)	Thermal Conductivity (W/m·K)	Thermal Diffusivity (m²/s*10?⁶)	Damping Depth (cm)
Air	20?C, Still	0.0012	1.01	0.0012	0.025	20.5	75.1
Water	20?C, Still	1.00	4.18	4.18	0.57	0.14	6.2
Clay Soil	Dry	1.60	0.89	1.42	0.25	0.18	7.1
(40% pore space)	Saturated	2.00	1.55	3.10	1.58	0.51	11.8

The heat capacity, C, of a substance is defined as the product of the density, ?, of the substance and the specific heat, c, of that substance. This relationship is demonstrated by the following equation:

$$? c = C \quad \text{(Eq. 1)}$$

? has units of kg/m3 , *c* is expressed in the units J/kg.K, and C has the units J/m3.K. Through motionless fluids and solids, heat gets transferred through the process of conduction, which is defined as energy being passed on from molecule to molecule in the substance. The rate of heat exchange has been found to be proportional to the temperature gradient in the direction of heat flow. The temperature gradient is defined as the temperature difference between a given distance in a substance. This relationship can be summarized by the following equation:

$$H = ?k(?T/?z) \quad \text{(Eq. 2)}$$

where H is the heat, T is the temperature, k is the thermal conductivity (expressed in the units of W/m.K), and z is the distance along the arbitrarily defined z-axis. This equation along with Eq. 1 can be expressed in the following form:

$$H/?c = ??h(?T/?z) \quad \text{(Eq. 3)}$$

where ?h is the thermal diffusivity, which is defined as:

$$?h = k/?c = k/C \quad \text{(Eq. 4)}$$

The thermal diffusivity, ?h, has units of m2/s. If the wave propagation is considered in one direction, namely from the surface downward or vice-versa, the rate of heat flowing within a given volume should be equal to the rate of change in the given volume. This can be assumed because of the law of conservation of energy. This equality can be expressed by the following equation:

$$(?/?t)(Cs\ T) = ??H/?z) \quad \text{(Eq. 5)}$$

where Cs is the heat capacity of the soil, and t is time.

If the heat capacity can be held constant with relation to time, we can substitute from Eq. 3 and obtain the following relationship:

$$?T/?t = (??/?z)[(k/Cs)(?T/?z)] = (??/?z)[?h(?T/?z)] \quad \text{(Eq. 6)}$$

With the integration of Eq. 5 we obtain:

$$H_G = H_D + ?_0^D ?/?t(CsT)\, dz \quad \text{(Eq. 7)}$$

where H_G is the ground heat flux, H_D is the soil heat flux, and D is a reference depth where H_D is equal to zero. The numerical solution to Eq. 7 can be expressed by the following equation:

$$T = T_m + A_s^{(?z/d)} \sin[(2?/P)(t-t_m)?z/d)] \quad \text{(Eq. 8)}$$

where T_m is the mean temperature of the subsurface, A_s is the amplitude of the thermal wave, P is the period of the thermal wave, and d is the damping depth, defined by the following equation:

$$d = (?h/?) \quad \text{(Eq. 9)}$$

The estimation of the predicted values of the temperature uses Eq. 8. The measurements of temperature considered in this model were taken at differing depths. These depths are respectively 5cm, 10cm, and 20cm below the surface.

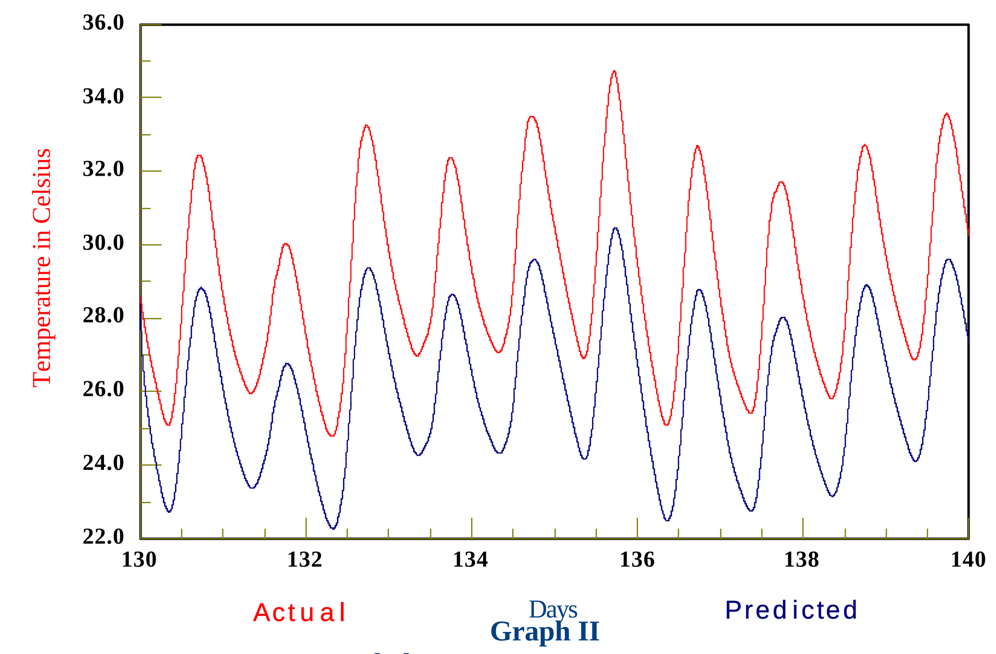
Procedure

The first obstacle to overcome during the research was to find an appropriate method to predict the middle 10cm level soil temperature. Using Fortran 77, a program was built that could analyze data from a pre-existent meteorological data file. The soil temperature data on the meteorological file was measured in 15-minute intervals. In order to feed more data to the integrating program, the data was linearly interpolated into 90-second intervals. Because 15 minutes is equal to 90 seconds, 9 linear approximations were made between each temperature measurement. The Fortran 77 program extracted and interpolated the appropriate soil temperature data.

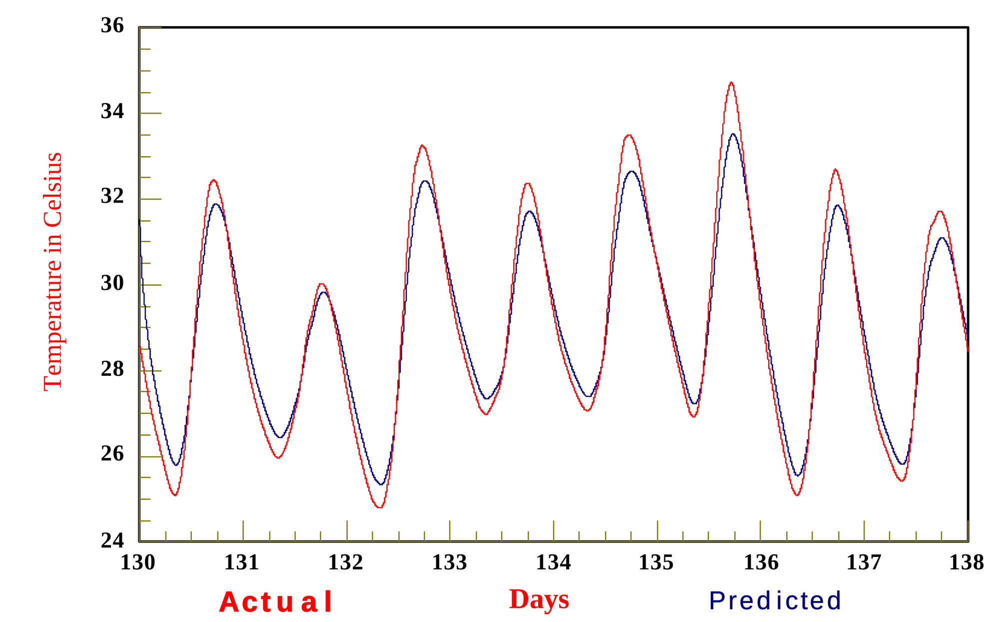
Another Fortran 77 program was built to predict the 10 cm level temperatures and compare them to the actual values. This program used Eq. 8 to predict several intermediate levels. The criterion for the accuracy of the prediction was selected. The criterion used was the sum of the squares of the differences between the actual and the predicted temperatures. The squares were used in order to negate the effect of a negatively signed difference and also to exaggerate higher errors comparatively. The program was written so that the damping depth, therefore the thermal diffusivity, could be varied. Using this method it was proposed that a close approximation of the actual damping depth could be made without having to use density, specific heat, and thermal conductivity measurements.

The days first selected for this comparison were selected because of the lack of rainfall during these days and similarities of the amount of solar radiation measured during these days. This was done in order to ensure similar moisture content and no abrupt surface temperature changes due to cloud cover. The graph of data analyzed from the selected days had a similar shape, but the predicted graph had values averaging about 3?C below the actual measured temperature. When the average difference between the predicted values and the actual values was added to the predicted values, the predicted values aligned almost perfectly with the actual values. This observation is illustrated by Graphs I and II. The comparison was performed on data selected from different periods and similar observations were noticed.

Graph I
Bledsoe Farm 2000 Days 130-140



Graph II
Bledsoe Farm 2000 Days 130-138



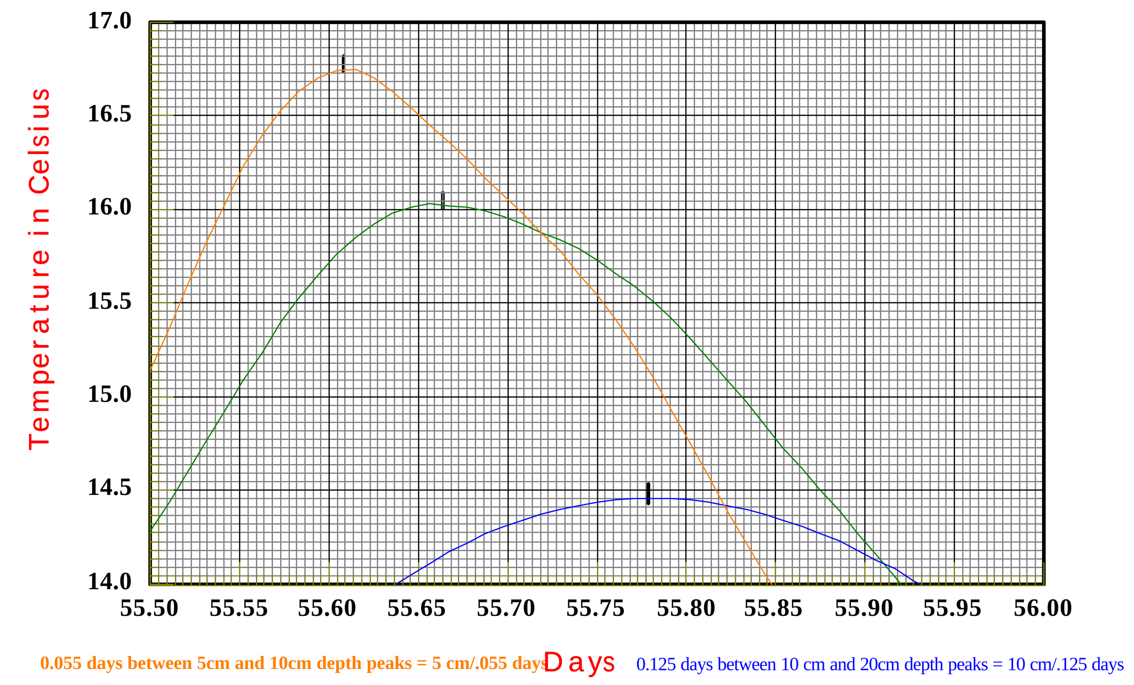
* Note that amplitude is a function of the damping depth and damping depth is not ideal in this case. The amplitude has been shifted +3.06 degrees.

The observations made on the predicted value led to the conclusion that some systematic error was involved. Again data was selected from the meteorological file that had no rainfall and therefore should minimize the possibility that different levels of soil had differing thermal diffusivities due to varying moisture content. These waveforms were analyzed graphically to see if the thermal waves between the three levels measured were traveling at a constant velocity. It was discovered that from the thermal waves observed that the thermal waves were not being recorded at uniform velocities. This indicated that the sensors' reported depth might not be accurate. There were two possibilities that could be tested. The possibilities were that two of the three sensors were buried at the appropriate depths and one sensor was not. The two correct sensors could either be assumed to be the 5 cm and the 10 cm sensors, or the 10 cm and the 20 cm sensors. Assuming that the 5 cm and 10 cm sensors were the correct distance apart, a distance of 11.3 cm between the 10cm and "20 cm" sensors could be assumed by the cross multiplication of distance to time (wave velocity). If the 10 cm and 20 cm sensors were assumed to be the correct distance apart, the "5 cm" sensor, by the same token, must be approximately 4.4 cm away from the 10 cm sensor. Both scenarios were tried and the latter scenario was found to give very close results. This process is demonstrated in Graph III and Table II.

Table II.

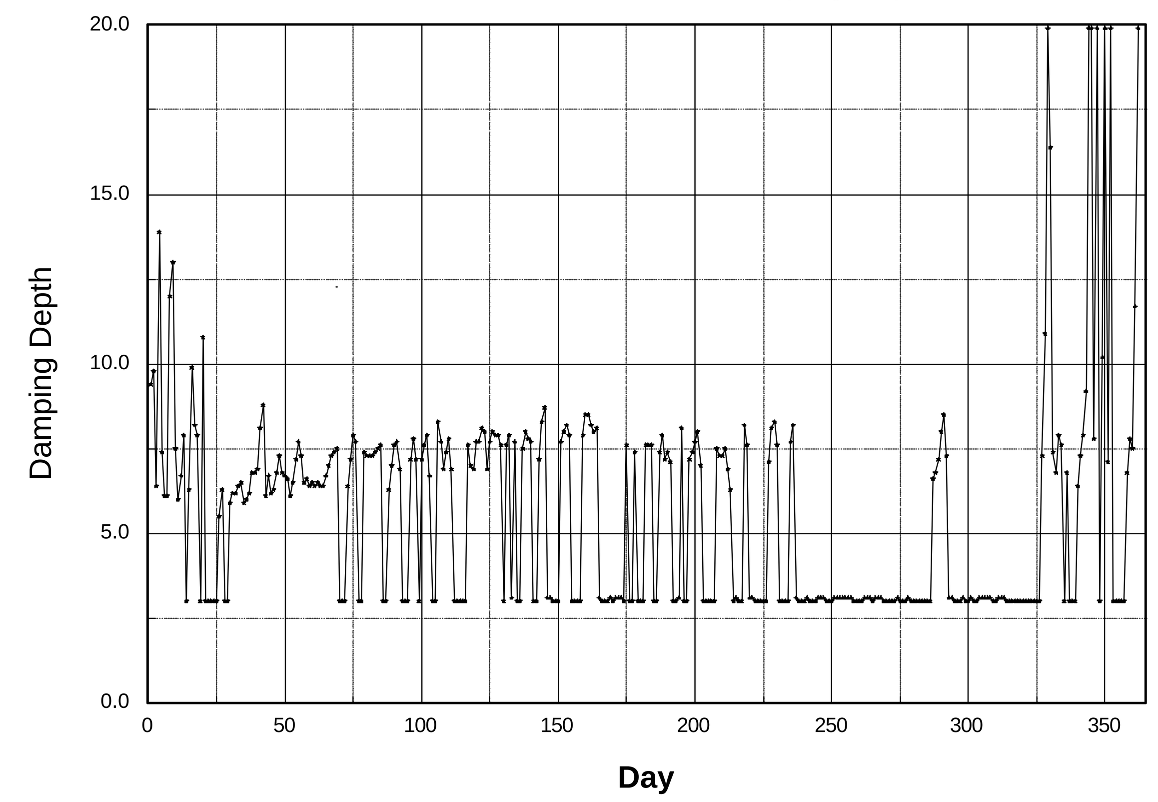
Depths assumed to be the Correct Distance Apart	Distance Between Levels	Time in Days Between Arriving Wavefronts	Assumed Correct Wave Speed	Distance of Incorrect level from 10 cm Depth
5 cm - 10 cm	5 cm	0.055	90.9 cm/day	11.36 cm
10 cm to 20 cm	10 cm	0.125	80.0 cm/day	4.4 cm

Graph III
Thermal Wave Speeds

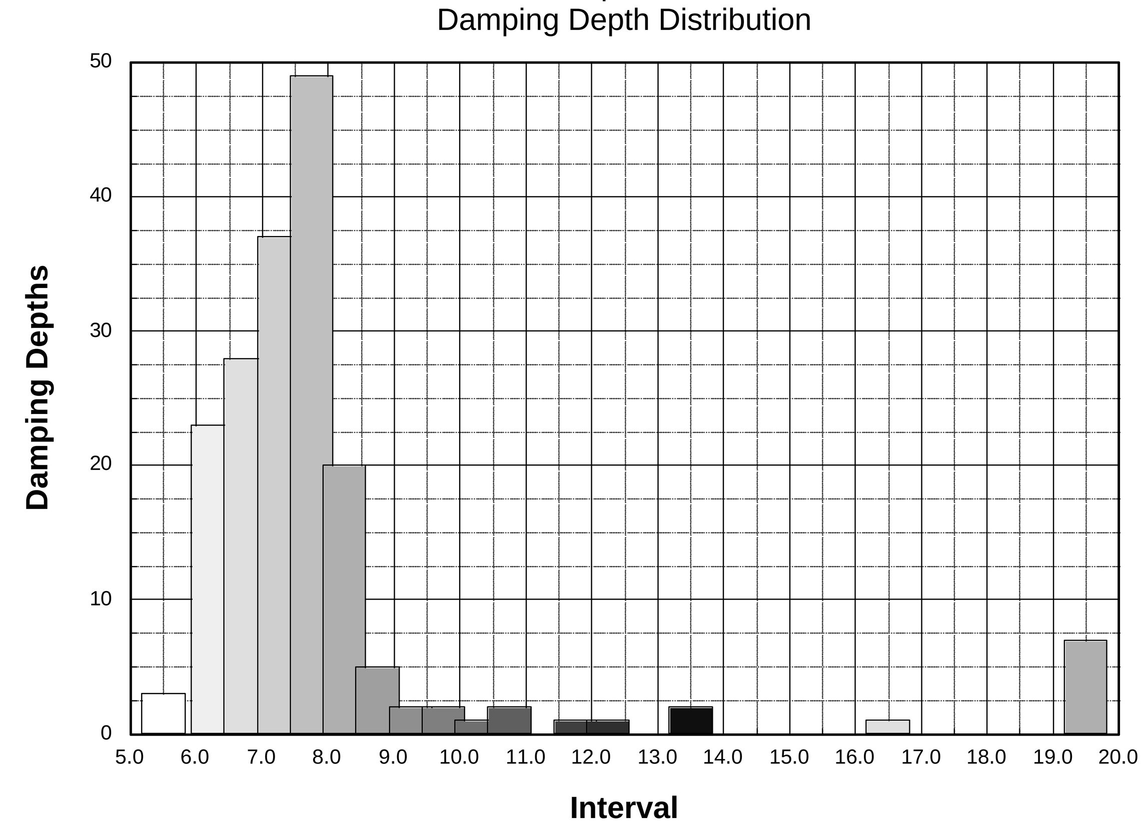


Next the two Fortran 77 programs were combined in order to create a program that would evaluate the damping depth for an entire year of data. This program calculated the daily damping depth for a year. The program again used the error between the predicted and actual values to choose the damping depth that caused the best fit, therefore the lowest error. The damping depth was started at 3.0 cm and increased by 0.1 cm until it reached 20 .0 cm. The results of this evaluation for the year 2000 are illustrated in Graph IV. A histogram of the damping depths is found in Graph V. The damping depths which have physical significance mostly fall into a range that is predicted by the Georgia clay soil type.

Graph IV
Bledsoe Farm 2000 Best Fit Damping Depths



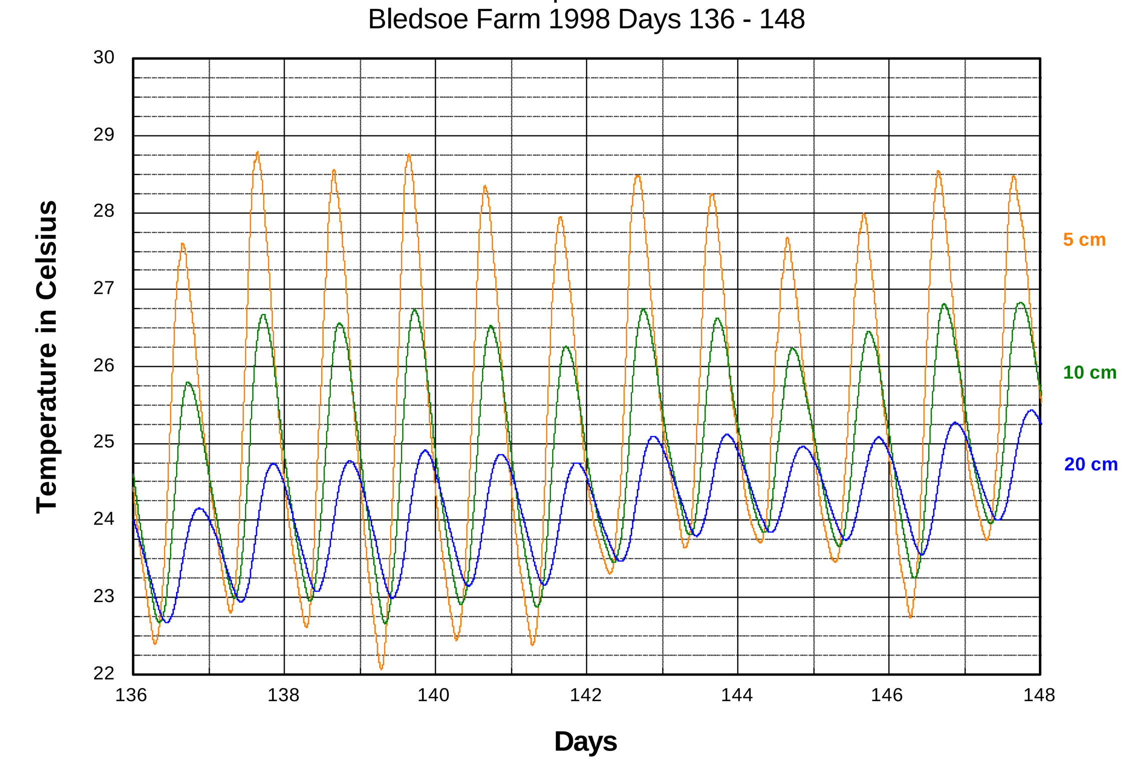
Graph V
Damping Depth Distribution



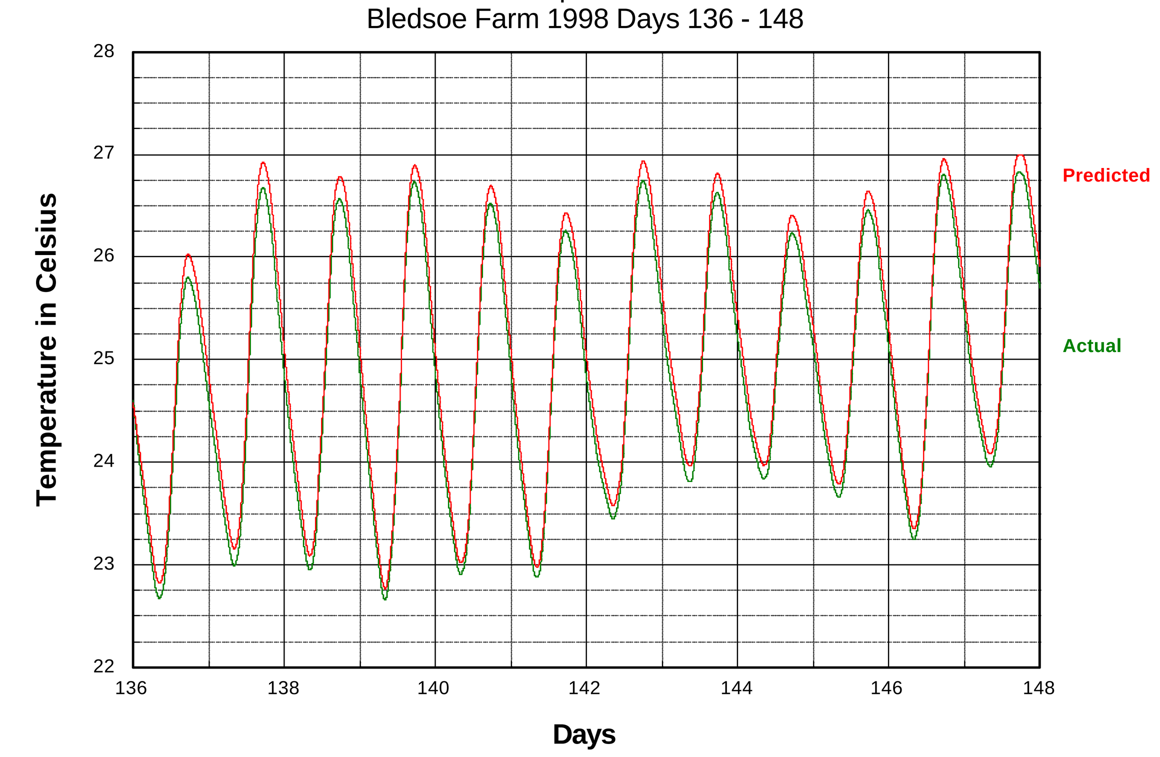
The damping depths that approach zero are caused by error at the beginning of the function where the initial temperatures are approximated and the function matching criterion is not able to recover. Additional error is also thrown into the evaluation because occasionally the 10 cm depth temperature is physically higher than both the 5 cm and the 20 cm depth temperatures. This model, having only three physical measurements, can only approximate the mid level temperature if its wave's peak is in between the layer's temperatures it is sandwiched between. This problem could possibly be resolved by placing additional sensors buried at varying depths.

Next, data was chosen from the Bledsoe Farm meteorological data file that could give the most accurate prediction. This data was chosen because of the lack of rainfall before and during the selected period. It was also chosen because the mid-level temperature peaks are located between the 5 cm and 20 cm depth temperatures. This sandwiching is illustrated in Graph VI. As can be seen from Graph VII., a damping depth of 8.6 cm was the model's best-fit and produced a fairly accurate prediction for the time period chosen. This best-fit damping depth falls well within the range predicted by the Georgia clay soil type (7.1-11.8 cm) and is also indicative of dryer soil that would be expected after a period of no rainfall. The model was tested at other similar periods and showed similar results.

Graph VI
Bledsoe Farm 1998 Days 136 - 148



Graph VII.
Bledsoe Farm 1998 Days 136 - 148



Conclusion

This model is limited slightly by its inability to predict temperature peaks that are not sandwiched by the boundary levels, however could still be used as a more accurate model than the existing ones. The research could also prove valuable as method of remotely typing soil by predicting the damping depth range that the soil exhibits. The method of thermal wave speed analysis could also be of use in detecting sensor placement inaccuracies and allowing for that correction to be made in the current and future models. Further research might include burying more sensors for models that could handle non-sandwiched peaks. Also an actual damping depth could be physically measured during a period ideal for model prediction. The predicted and actual damping depth could then be compared.

References.

1. Arya, S. Pal, 2001, Introduction to Micrometeorology, 2nd Edition, Academic Press, London.